Final Exam — Functional Analysis (WBMA033-05)

Friday 5 April 2024, 11.45–13.45h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

The linear space $\mathcal{C}([0,1],\mathbb{K})$ can be equipped with the following norms:

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Are these norms equivalent? Motivate your answer.

Problem 2 (5 + 5 + (5 + 10) = 25 points)

Consider the following Banach space over $\mathbb{C}$:

$$X = \left\{ f : \mathbb{R} \to \mathbb{C} \,:\, \sup_{x\in\mathbb{R}} |f(x)| < \infty \right\}, \quad \|f\| = \sup_{x\in\mathbb{R}} |f(x)|.$$

For a fixed constant $\tau > 0$ consider the following linear operator:

$$T : X \to X, \quad Tf(x) = f(x-\tau).$$

(a) Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $|\lambda| = 1$.

(b) Show that every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ is an eigenvalue of T. Hint: complex exponentials.

(c) Show in two different ways that T is *not* compact:

 (i) By using properties of $\sigma(T)$.

 (ii) By considering the sequence (Tf_n) for a suitably chosen sequence (f_n) in X.

Problem 3 (8 + 7 = 15 points)

Let X be a Hilbert space over $\mathbb{C}$. Assume $U, V \in B(X)$ are selfadjoint and $UV = VU$. Define the operator $T \in B(X)$ by $T = U + iV$.

(a) Show that $\|Tx\|^2 = \|Ux\|^2 + \|Vx\|^2$ for all $x \in X$.

(b) Show that for $V = I$ the operator T is injective and has a closed range.

Turn page for problems 4 and 5!

Problem 4 (5 + (4 + 6) = 15 points)

(a) Formulate the uniform boundedness principle.

(b) Let X be a Hilbert space over $\mathbb{K}$ and let $V \subset X$ be a nonempty subset.

(i) For a fixed $v \in V$ define the linear map $f_v : X \to \mathbb{K}$ by $f_v(x) = \langle x, v \rangle$. Show that $\|f_v\| = \|v\|$.

(ii) Assume that for each $x \in X$ there exists a constant $M_x \geq 0$ such that

$$|\langle v, x \rangle| \leq M_x \quad \text{for all } v \in V.$$

Prove that the set V is bounded.

Problem 5 (12 + 5 + 8 = 25 points)

Let X be a normed linear space over $\mathbb{K}$ and assume that the vectors $x_1, x_2 \in X$ are linearly independent.

(a) Prove that the map $(a_1, a_2) \mapsto \|a_1 x_1 + a_2 x_2\|$ is a norm on $\mathbb{K}^2$.

(b) Prove that there exists a constant $C > 0$ such that

$$|3a_1 - 5a_2| \leq C\|a_1 x_1 + a_2 x_2\| \quad \text{for all} \quad (a_1, a_2) \in \mathbb{K}^2.$$

(c) Prove that there exists $f \in X'$ such that

$$f(x_1) = 3, \quad f(x_2) = -5, \quad \|f\| \leq C.$$

End of test (90 points)

Solution of problem 1 (10 points)

If the two norms are equivalent, then there exist constants $0 < m \leq M$ such that

$$m\|f\|_1 \leq \|f\|_\infty \leq M\|f\|_1$$

for all $f \in \mathcal{C}([0,1],\mathbb{K})$.
(3 points)

Consider the sequence in $\mathcal{C}([0,1],\mathbb{K})$ given by $f_n(x) = x^n$. For all $n \in \mathbb{N}$ we have

$$\|f\|_1 = \frac{1}{n+1} \quad \text{and} \quad \|f\|_\infty = 1.$$

(3 points)

In particular, for all $n \in \mathbb{N}$ we have

$$1 \leq \frac{M}{n+1}.$$

(3 points)

Taking $n \to \infty$ gives $1 \leq 0$, which is obviously a contradiction. Therefore, the norms are not equivalent.
(1 point)

Solution of problem 2 (5 + 5 + (5 + 10) = 25 points)

(a) For all $f \in X$ we have

$$\|Tf\| = \sup_{x\in\mathbb{R}} |(Tf)(x)| = \sup_{x\in\mathbb{R}} |f(x-\tau)| = \sup_{x\in\mathbb{R}} |f(x)| = \|f\|.$$

(3 points)

If $Tf = \lambda f$ for some nonzero $f \in X$, then

$$\|f\| = \|Tf\| = \|\lambda f\| = |\lambda|\,\|f\|.$$

Dividing both sides by $\|f\|$ gives $|\lambda| = 1$.
(2 points)

(b) If $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$, then for some $\theta \in \mathbb{R}$ we have $\lambda = e^{i\theta}$. Consider the (nonzero!) function $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{-i\theta x/\tau}$. Then we have

$$(Tf)(x) = f(x-\tau) = e^{-i\theta(x-\tau)/\tau} = e^{i\theta}e^{-i\theta x/\tau}.$$

In other words, we have $Tf = \lambda f$.
(5 points)

(c) (i) There are (at least) three ways we can use the spectrum to show that T is not compact.

Method 1. If T is compact, then we can use a theorem that states that $0 \in \sigma(T)$ when $\dim X = \infty$. However, the operator $S : X \to X$ given by $(Sf)(x) = f(x+\tau)$ satisfies $ST = TS = I$ and is bounded. Therefore, T is invertible and thus $0 \in \rho(T)$. Therefore, T cannot be compact.
(5 points)

Method 2. In part (b) we have shown that $\lambda = 1$ is an eigenvalue of T. The corresponding eigenspace consists of all τ-periodic functions and thus is infinite-dimensional. Indeed, for all $n \in \mathbb{N}$ the functions $f_n(x) = \sin(2n\pi x/\tau)$ are eigenfunctions and these functions are linearly independent.
(3 points)

However, compact operators have the property that eigenspaces corresponding to nonzero eigenvalues must be finite-dimensional. Therefore, T cannot be compact.
(2 points)

Method 3. If T is compact, then we can use a theorem that states that T can only have countably many eigenvalues. However, from part (b) it follows that T has uncountably many eigenvalues. Therefore, T cannot be compact.
(5 points)

(ii) For each $n \in \mathbb{N}$ define the following function:

$$f_n(x) = \begin{cases} 1 & \text{if } x = n\tau, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence (f_n) belongs to X and is bounded as $\|f_n\| = 1$ for all $n \in \mathbb{N}$.
(4 points)

But if $n \neq m$ we have $\|Tf_n - Tf_m\| = \|f_{n+1} - f_{m+1}\| = 1$.
(3 points)

Therefore, the sequence (Tf_n) does not have a convergent subsequence. We conclude that T cannot be compact.
(3 points)

Remark. There are many possible examples that can be constructed in this way. Take any function $f \in X$ that vanishes outside an interval of length τ. Then define the sequence (f_n) in X by $f_n = T^n f$ and proceed as above.

Solution of problem 3 (8 + 7 = 15 points)

(a) The adjoint of T is given by

$$T^* = (U + iV)^* = U^* + (iV)^* = U - iV.$$

(2 points)

This gives

$$T^*T = (U - iV)(U + iV) = U^2 + i(UV - VU) + V^2 = U^2 + V^2.$$

(2 points)

Finally, using that U and V are selfadjoint, gives

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle x, T^*Tx\rangle \\ &= \langle x, (U^2 + V^2)x\rangle \\ &= \langle x, U^2x\rangle + \langle x, V^2x\rangle \\ &= \langle Ux, Ux\rangle + \langle Vx, Vx\rangle \\ &= \|Ux\|^2 + \|Vx\|^2.\end{aligned}$$

(4 points)

(b) *Method 1.* If $V = I$, then part (a) gives $\|Tx\|^2 = \|Ux\|^2 + \|x\|^2 \geq \|x\|^2$ and thus $\|Tx\| \geq \|x\|$ for all $x \in X$.
(1 point)

Recall the Closed Range Theorem, which states the following: if X and Y are Banach spaces and $T \in B(X, Y)$, then the following statements are equivalent:

(i) There exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$.

(ii) The operator T is injective and its range is closed.

(4 points)

We can then apply this theorem with $c = 1$.
(2 points)

Method 2. If $V = I$, then part (a) gives $\|Tx\|^2 = \|Ux\|^2 + \|x\|^2 \geq \|x\|^2$ and thus $\|Tx\| \geq \|x\|$ for all $x \in X$.
(1 point)

If $Tx = 0$, then $x = 0$ which shows that T is injective.
(1 point)

Let y_n be a sequence in $\operatorname{ran} T$ such that $y_n \to y$. There exists a sequence (x_n) in X such that $y_n = Tx_n$ for all $n \in \mathbb{N}$. Note that since (y_n) is Cauchy, so is $\|x_n\|$ in view of the inequality we have for T. But then $x_n \to x$ for some $x \in X$ since X is complete. Finally, since $T \in B(X)$ we have $Tx_n \to Tx$. By uniqueness of limits we conclude that $y = Tx \in \operatorname{ran} T$ and thus that $\operatorname{ran} T$ is closed.
(5 points)

Solution of problem 4 (5 + (4 + 6) = 15 points)

(a) There are (at least) two formulations that can be given.

Formulation 1. Let X be a Banach space and let Y be a normed linear space. Let $F \subset B(X, Y)$ and assume that

$$\sup_{T \in F} \|Tx\| < \infty \quad \text{for all} \quad x \in X.$$

Then the elements $T \in F$ are uniformly bounded:

$$\sup_{T \in F} \|T\| < \infty.$$

(5 points)

Formulation 2. Let X be a Banach space and let Y be a normed linear space. Let $F \subset B(X, Y)$ and assume that

$$M = \left\{x \in X \,:\, \sup_{T \in F} \|Tx\| < \infty\right\}$$

is nonmeager. Then the elements $T \in F$ are uniformly bounded:

$$\sup_{T \in F} \|T\| < \infty.$$

(5 points)

(b) (i) For $x \in X$ the Cauchy-Schwarz inequality gives $|f_v(x)| = |\langle x, v\rangle| \leq \|x\|\|v\|$, which implies that

$$\sup_{x \neq 0} \frac{|f_v(x)|}{\|x\|} \leq \|v\|.$$

(3 points)

For $x = v$ we have

$$\frac{|f_v(x)|}{\|x\|} = \frac{|\langle v, v\rangle|}{\|v\|} = \|v\|.$$

Hence, $\|f_v\| = \|v\|$.
(1 point)

(ii) For any $x \in X$ there exists a constant $M_x \geq 0$ such that

$$|f_v(x)| = |\langle x, v\rangle| = |\langle v, x\rangle| \leq M_x,$$

which implies that

$$\sup_{v \in V} |f_v(x)| < \infty \quad \text{for all } x \in X.$$

(3 points)

By part (a) and the uniform boundedness principle we have

$$\sup_{v \in V} \|v\| = \sup_{v \in V} \|f_v\| < \infty,$$

which implies that the set V is bounded.
(3 points)

Solution of problem 5 (12 + 5 + 8 = 25 points)

(a) Write $\|(a_1, a_2)\|_* = \|a_1x_1 + a_2x_2\|$. We have $\|(a_1, a_2)\|_* \geq 0$ for all $(a_1, a_2) \in \mathbb{K}^2$.
(1 point)

Since $\|\cdot\|$ is a norm on X and x_1 and x_2 are linearly independent, it follows that

$$\begin{aligned} \|(a_1, a_2)\|_* = 0 \quad &\Rightarrow \quad \|a_1x_1 + a_2x_2\| = 0 \\ &\Rightarrow \quad a_1x_1 + a_2x_2 = 0 \\ &\Rightarrow \quad a_1 = a_2 = 0. \end{aligned}$$

(3 points)

For any $\lambda \in \mathbb{K}$ and $(a_1, a_2) \in \mathbb{K}^2$ we have

$$\begin{aligned} \|\lambda(a_1, a_2)\|_* &= \|(\lambda a_1, \lambda a_2)\|_* \\ &= \|\lambda a_1x_1 + \lambda a_2x_2\| \\ &= |\lambda|\, \|a_1x_1 + a_2x_2\| \\ &= |\lambda|\, \|(a_1, a_2)\|_*. \end{aligned}$$

(4 points)

For any $(a_1, a_2), (b_1, b_2) \in \mathbb{K}^2$ we have

$$\begin{aligned} \|(a_1, a_2) + (b_1, b_2)\|_* &= \|(a_1 + b_1, a_2 + b_2)\|_* \\ &= \|(a_1 + b_1)x_1 + (a_2 + b_2)x_2\| \\ &= \|(a_1x_1 + a_2x_2) + (b_1x_1 + b_2x_2)\| \\ &\leq \|a_1x_1 + a_2x_2\| + \|b_1x_1 + b_2x_2\| \\ &= \|(a_1, a_2)\|_* + \|(b_1, b_2)\|_*. \end{aligned}$$

(4 points)

(b) There are at least two approaches that can be taken.

Method 1. Consider the linear subspace $V = \operatorname{span}\{x_1, x_2\}$ and define the linear map

$$f : V \to \mathbb{K}, \quad f(a_1x_1 + a_2x_2) = 3a_1 - 5a_2,$$

Since linear maps between two finite-dimensional spaces are bounded there exists a constant $C > 0$ such that

$$|3a_1 - 5a_2| = |f(a_1x_1 + a_2x_2)| \leq C\|a_1x_1 + a_2x_2\|$$

for all $(a_1, a_2) \in \mathbb{K}^2$.
(5 points)

Method 2. We have $|3a_1 - 5a_2| \leq 3|a_1| + 5|a_2| \leq 5(|a_1| + |a_2|)$.
(1 point)

Note that $\|(a_1, a_2)\|_1 = |a_1| + |a_2|$ is also a norm on $\mathbb{K}^2$.
(1 point)

Since all norms on a finite-dimensional space are equivalent, there exists a constants $m, M > 0$ such that

$$m\|(a_1, a_2)\|_* \le \|(a_1, a_2)\|_1 \le M\|(a_1, a_2)\|_*$$

for all $(a_1, a_2) \in \mathbb{K}^2$. Finally, let $C = 5M$.
(3 points)

(c) Consider the linear subspace $V = \operatorname{span}\{x_1, x_2\}$ and define the linear map

$$f : V \to \mathbb{K}, \quad f(a_1x_1 + a_2x_2) = 3a_1 - 5a_2,$$

so that $f(x_1) = 3$ and $f(x_2) = -5$.
(2 points)

In addition, by part (b) we have for all $(a_1, a_2) \in \mathbb{K}^2$ that

$$|f(a_1x_1 + a_2x_2)| = |3a_1 - 5a_2| \le C\|a_1x_2 + a_2x_2\|.$$

This implies that

$$\|f\|_{V'} = \sup_{v \in V\setminus\{0\}} \frac{|f(v)|}{\|v\|} \le C.$$

(6 points)

By the Hahn-Banach theorem there exists an extension of f (again denoted by f) to the entire space X which preserves the operator norm: $\|f\|_{X'} = \|f\|_{V'} \le C$.
(2 points)